A Guide to the Technical Finance Literature on Uncertainty Aversion
James Juniper
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A Liquidity Preference Theory of Market Prices

In the introduction to the first volume of his collected papers Boulding refers to paper No. 9 as, “I think, perhaps I would nominate as my most important contribution to economics (although, again, one to which no one has paid much attention). This note sets out Boulding’s analysis in this paper focusing primarily on the price determination component. It also looks at how his model can readily be modified to account for an endogenous-money economy subject to interest rate targeting. This analysis follows a precedent set by Randall-Wray (1991).

Let the demand-supply curve of individual $i$ depicted above (in linear form) be given by:

$$ p_i = b_i q_i + c_i $$

BUY

SELL

$f_i(p) > 0$ Null price $f_i(p) < 0$

Let the demand-supply curve of individual $i$ depicted above (in linear form) be given by:

$$ q_i = f_i(p) $$

Here a negative value would represent an offer to sell and a positive value an offer to buy. Assume the $i$th individual posses an amount of money $m_i$, an endowment of corn (in a one good economy) $a_i$, and a desired ratio of money to the total of valuta $r_i$ (assumed to be independent of $p_i$), and let the amount of the good exchanged by that individual equal $q_i$. Thus, after the transaction the individual would possess $q_i + a_i$ of the commodity with value $p_i q_i + p_i a_i$, and $m_i - p_i q_i$ money so that the total value of his holding of both money and the commodity would be $m_i + p_i a_i$. Thus the liquidity ratio would be:

$$ r_i = \frac{m_i - p_i q_i}{m_i + p_i a_i} $$

Accordingly (solving for the price):

$$ p_i = \frac{m_i (1 - r_i)}{r_i a_i + q_i} $$

Solving for $q_i$ gives:
\[ q_i = \frac{m_i(1-r_i)}{p_i} - r_i a_i, \]
which is a rectangular hyperbola with asymptotes:
\[ q_i = \begin{cases} \infty \\ -r_i a_i \end{cases}. \]
In a market equilibrium for all \( N \) traders:
\[ Q_e = Q_d - Q_s = \sum_{i=1}^{N} f_i = 0 \]
where:
\[ Q_d = \sum_{i=1}^{N} f_i|f_i| < 0 \]
\[ Q_s = \sum_{i=k+1}^{N} f_i|f_i| > 0 \]
The substitution of the expression for \( q_i \) into this market clearing condition yields:
\[ p = \frac{m_1(1-r_1) + \cdots + m_N(1-r_N)}{r_0 a_1 + \cdots + r_N a_N}. \]
Thus \( p \) rises if the \( m \)s rise, the \( a \)s fall, or the \( r \)s fall; and the affect of a change in \( m \) on \( p \) is heightened if \( r \) is higher. Boulding goes on to show how this equation can be used to derive an expression for the quantity of the good actually traded, which he shows is responsive to the diversity of null or reservation prices across the population of traders, although he concedes that the resulting expression is less useful than it might at first appear because there can be no clear distinction in the market between buyers and sellers.

If the preferred liquidity ratio of all individuals is the same and equal to \( r \), the price equation reduces to (p. 140):
\[ p = \frac{M(1-r)}{Ar}, \]
where \( A \) is the total stock of the commodity, and \( M \) is the total stock of money. It follows that an increase in the quantity of money, or a decrease in liquidity preference, will raise all prices (p. 141). Boulding notes that this equation can be compared with Fisher’s equation of exchange (p. 140). He calls the ratio \( (1-r)/r \) the ‘velocity ratio’ because it rises and falls with the velocity of circulation of money as \( r \) falls and rises. However, while Fisher’s equation describes an identity of flows, Boulding notes that his equation describes an identity of stocks. Moreover, although Fisher’s equation is unable to distinguish between changes in prices and changes in the volume of transactions, his own equation can separately determine changes in the price level.

This analysis can readily be generalized to the case of a multiplicity of \( Z \) commodities (and non-money assets) \( A, B, C, D,\ldots, Z \). Following Boulding, let \( p_a \) be the price of commodity \( A \) and let \( A \) be the amount held by traders with the total value of all commodities (including non-money assets) and money be \( v \). The commodity preference ratio for commodity \( A \) can now be written as:
\[ r_a = \frac{A p_a}{v}, \]
while the liquidity preference ratio is given by:
\[ r_m = \frac{M}{v} \]

Eliminating \( v \) in both of these equations gives:

\[ p_a = \frac{Mr_a}{Ar_m}. \]

A similar equation can be derived for each of the other commodities (and non-money assets). Boulding notes that his analysis “enables us to see the so-called ‘liquidity preference’ theory of interest as merely a special case of (the above) equation” (p. 142). Given that the interest rate is inversely related to the price of a fixed interest security, \( p_s \) he notes that the determination of a present price for such a security is equivalent to targeting the expected rate of interest as in the equation:

\[ p_s = \frac{Mr_s}{Sr_m}. \]

This equation can readily be transformed to accommodate endogeneity of the money supply:

\[ M = \frac{p_s Sr_m}{r_s}, \]

as a function of the chosen security price (which could be the overnight cash rate for example). This equation can now be substituted back into the equation for a particular commodity (or for the general price level) to yield:

\[ p_a = \frac{p_s Mr_m r_a}{Ar_m r_s} = \frac{p_s Sr_a}{Ar_s}, \]

which shows that, for a given security price target, the price of the commodity is a positive (negative) function of the commodity (security) preference ratio and of the rate of interest on the security (endowment of the commodity). This is essentially the same approach that Randall Wray adopts to what he describes as “Boulding’s Balloons” in a respectful but concise re-working of Boulding’s analysis using the abbreviative power of matrix notation.

Here the \( R_{Ai} \)'s = desired asset ratios, \( R_m = \) liquidity preference (i.e. The desired asset ratio of money), the \( A_i = \) quantities of each asset, and the \( P_{Ai} = \) respective asset prices. When many forms of money exist the multiplicative component in front of the matrix in the above expression for Boulding’s identity can be expanded as shown in the first equation in the following diagram. The next three equations identify the cross-equation restrictions which must hold in the extended version of the asset pricing identity:

\[
\begin{bmatrix}
P_{A1} \\
P_{A2} \\
\vdots \\
P_{An}
\end{bmatrix} = \begin{bmatrix}
1 \\
0 \\
\vdots \\
0
\end{bmatrix} \begin{bmatrix}
R_{A1} & 0 & \cdots & 0 \\
0 & R_{A2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & R_{An}
\end{bmatrix} \begin{bmatrix}
1 \\
A_1 \\
1 \\
A_2 \\
\vdots \\
1 \\
A_n
\end{bmatrix}
\]

\[ R_m + \sum_{i=1}^{n} R_{Ai} = 1 \]

\[ \Rightarrow P_{Ai} = \left( \frac{1}{R_m} \right) M \begin{bmatrix}
R_{Ai} & \frac{1}{A_i}
\end{bmatrix} \]

Here the \( R_{Ai} = \) desired asset ratios, \( R_m = \) liquidity preference (i.e. The desired asset ratio of money), the \( A_i = \) quantities of each asset, and the \( P_{Ai} = \) respective asset prices. When many forms of money exist the multiplicative component in front of the matrix in the above expression for Boulding’s identity can be expanded as shown in the first equation in the following diagram. The next three equations identify the cross-equation restrictions which must hold in the extended version of the asset pricing identity:
Here $M_1$ (high powered money) is dependent on the purchase of $A_1$ by the Central Bank; $M_2-M_4$ (bank money, including demand deposits and other chequable liabilities) is dependent on purchases of assets $A_2-A_4$ (liabilities issued by borrowers including commercial paper and corporate bonds) by the commercial banks; and $M_5-M_k$ (non-bank financial institutions [NBFI] money) is dependent on purchase of assets $A_2-A_k$ by Non Bank Financial Institutions (here the $k$ subscript identifies purchases of assets by NBFI). Purchases of $A_1-A_k$ by the general public do not lead to any change in money supply.

A rise in liquidity preference implies that the left hand side $R_{mj}$'s are greater than the right hand side $R_{mj}$'s; the $R_{mj}$'s are greater than the $R_{mi}$'s; and the left hand side $R_{di}$'s are greater than the right hand side $R_{di}$'s. Hence liquidity preference has the greatest effect on the $P_A$'s in the lower portion of the $A$ vector. On one hand, interest on bank liabilities (money) determines the banking sector’s willingness to buy non-bank liabilities, while, on the other hand, changes in the upper portion of the $A$ vector determine fluctuations in the term structure of interest rates.

Wray uses this identity to clarify Keynes’ properties of money, namely: 1) a small elasticity of production, which implies that an increase in liquidity preference does not lead to diversion of labour into money production; instead, interest rates rise because a higher value for $R_m$ implies that $P_A$ falls and preferences for assets in the lower right hand side of the $R_d$ matrix which require high labour inputs, also fall; 2) a small elasticity of substitution, which is expressed in the fact that a rise in liquidity preference implies a higher demand for assets in upper portion of $A$ vector; hence $P_A$ is lower for assets in the lower portion of the matrix, but substitution will not occur because these assets can’t satisfy the desire for liquidity; and, 3) a high and positive liquidity premium, which is reflected in the fact that the return on money does not fall quickly as $M$ is increased, due primarily to its negligible carrying costs.

**Jump Diffusion Models**

Jump-diffusion models were developed to account for the combined influence of normal, high-frequency activity and the infrequent occurrence of extreme events (e.g., see Merton, 1976). Formally, this was accomplished by adding a Wiener diffusion processes representing high levels of normal activity to a Poisson jump processes representing rare and extreme events. In each case, Merton demonstrates that the volatility of the component process is proportional to duration. For the normal process this is because the size of the jump is proportional to duration although likelihood of a jump is constant. For the Poisson process, the size of the jump is constant while the likelihood of a jump increases with duration.
Rockinger and Jondeau (2002) instance other approaches to capturing the fat-tailed distributions that are observed in real financial markets including Bollerslev’s (1987) GARCH Model, in which the unconditional distribution is presumed to be the Student-$t$, the generalized error distribution (Nelson, 1991), the non-central gamma distribution (Harvey and Siddiquie, 1999) and the generalized Student-$t$ (Hansen, 1994), where the degree of freedom and asymmetry parameter are assumed to be time-varying. In their own paper they set out an entropy-density approach for estimating autoregressive conditional skewness and kurtosis based on potential theory and techniques of numerical integration.

Nelson (1990) shows that Hull and White’s (1988a,b) well known continuous-time, stochastic volatility model can be estimated by representing the drift process as autoregressive of order one, and the conditional volatility process as GARCH using a GARCH (1,1) process. More generally, it can be conjectured that the variance of the logarithmic rates of change in the asset price follows a Power ARCH scheme. Fornari and Mele (1997) combine such a Power ARCH model for the variance with an additional equation for the log relative price ratio, making the further assumption that the error term for this equation is drawn from a general error distribution.

**Time-Changed Brownian Motion**

Platen (1998) and Geman et al., (1998) oppose the jump-diffusion model on the grounds that it is ad-hoc and lacks a sound grounding in economic theory. Instead, these new models represent asset price realisations as the outcome of an equilibrium between supply and demand: i.e., as the difference of two increasing random processes triggering off up and down moves in the market occasioned, respectively, by the cumulative arrival of orders to buy and to sell. In turn, these buy and sell orders are viewed as spanning a continuum which stretches from high frequency orders of small incremental size to low frequency orders of larger incremental size. Some allowance can also be made for the plausible notion that price responses are less elastic for large increments than they are for small increments. In this way, price realisations are viewed within an integrated framework as resulting from a uniform response mechanism, irrespective of whether the arriving orders are of high frequency or low frequency.

The stochastic processes for asset prices generated in this class of model take the form of time-changed Brownian motion (ie., semi-martingale subordinators). They can be characterised as simple Brownian motion, only when the clock-time of the stochastic process is seen to be driven by the level of trading activity. Here, trading activity is viewed as being determined by two (correlated) sets of state variables: one representing the volume of buy and sell orders, and the other, the magnitude of orders. Other features include naturally arising stochastic volatility and stochastic interest rates.

This form of subordinator process gives rise to distribution functions of the generalised hyperbolic distribution class which can encompass a wide variety of distributions already extensively discussed in the literature, as cited by Platen (1998) and Geman et al., (1998); including the variance gamma (Madan & Seneta, 1990), the normal inverse gaussian (Barndorff-Nielson 1977, 1978, 1995), the Student-$t$ (Hurst and Platen 1997; Hurst, Platen and Rachev, 1997), and the hyperbolic distributions (Eberlein and Keller, 1995; Küchler et al., 1995). Geman et al, also cite work examining implied volatility smiles in options markets in this specific equilibrium context by Bates (1996), and Bakshi, Cao and Chen (1997).

Geman et al., consider the price process $p(t)$ for a single commodity trading continuously over the time interval $[0,T]$. The uncertainties driving the price process are conceived as cumulative demand and supply shocks given, at any time $t$, by the increasing pure jump processes $U(t)$ and $V(t)$, respectively, so that the amount of units of the asset demanded by agents in the economy consequent upon the occurrence of liquidity and information considerations or wealth and cash balance accumulation decisions are given by:
\[ u_t = \Delta U(t) = U(t) - U(t) \geq 0 \]
\[ v_t = \Delta V(t) = V(t) - V(t) \]

The actual quantity demanded in response to a demand shock is given by a demand function

\[ q_t^d = q_d\left(p(t)/p(t), u_t, t\right) = u_t - \delta \ln \left( \frac{p(t)}{p(t)} \right) \]

so that \( q_t^d(1, u_t, t) = 1 \), and \( \frac{\partial q_t^d}{\partial p(t)} < 0 \)

while the supply function response to a demand shock \( u_t \) is given by

\[ q_t^s = q_s\left(p(t)/p(t), u_t, t\right) = a_t u_t^{-\gamma}, \ln \left( \frac{p(t)}{p(t)} \right) \]

so that \( q_t^s(1, u_t, t) = 1 \), and \( \frac{\partial q_t^s}{\partial p(t)} > 0 \)

The market clearing condition can be solved simultaneously to determine the price response \( \Phi^p \) and quantity transacted \( \Psi^p \)

\[ q_t^s = q_t^d \Rightarrow \ln \left( \frac{p(t)}{p(t)} \right) = \Phi^p(1, u_t, t) > 0 = \frac{u_t}{\delta + a_t u_t^{-\gamma}} \]
\[ q_t^s = \Psi^p(1, u_t, t) > 0 = \frac{a_t u_t^{-\gamma}}{\delta + a_t u_t^{-\gamma}} \]

Similarly, the response of the supply function to a supply shock \( v_t \) is given be

\[ q_t^s = q_t^s \Rightarrow \ln \left( \frac{p(t)}{p(t)} \right) = \Psi^p(1, v_t, t) > 0 = \frac{v_t}{\eta + b_t v_t^{-\lambda}} \]

while the response of the demand function to the supply shock is given by

\[ q_t^d = q_t^d \Rightarrow \ln \left( \frac{p(t)}{p(t)} \right) = -b_t v_t^{-\lambda} \ln \left( \frac{p(t)}{p(t)} \right) \]

so that \( q_t^d(1, v_t, t) = 0 \), and \( \frac{\partial q_t^d}{\partial p(t)} < 0 \)

Once again, the market clearing assumption may be solved for the price response \( \Phi^v \) and quantity transacted \( \Psi^v \)
\[ q^*_t = q^{-}_t = q_t^\eta \Rightarrow \ln \left( \frac{p(t)}{p(t^-)} \right) = -\Phi^t(u, t) < 0 = -\left( \frac{v_t}{\eta + b_t v_t^\eta} \right) \]

\[ q^*_t = \Psi^t(v_t, t) = \frac{b v_t^{1-\lambda_t}}{\eta + b v_t^\lambda} \]

Under the additional assumption that demand shocks and supply shocks never arise at exactly the same instant of time, the resultant price process \( \ln p(t) \) and quantity transacted \( Q(t) \) are given by

\[
\ln p(t) = \ln p(0) + \sum_{s \leq t} \Phi^s(\Delta U(s), s) - \sum_{s \leq t} \Phi^s(\Delta V(s), s) \\
= \ln p(0) + \sum_{s \leq t} \frac{\Delta U(s)}{\delta_t + a_s (\Delta U(s))^\gamma_s} - \sum_{s \leq t} \frac{\Delta V(s)}{\eta_s + b_s (V(s))^\lambda_s}
\]

and

\[
Q(t) = \sum_{s \leq t} \Psi^s(\Delta U(s), s) + \sum_{s \leq t} \Psi^s(\Delta V(s), s) \\
= \sum_{s \leq t} \frac{a_s U(s)}{a_s + \delta_s (\Delta U(s))^\gamma_s} + \sum_{s \leq t} \frac{b_s V(s)}{b_s + \eta_s (V(s))^\lambda_s}
\]

When the supply responses to market order demand shocks and the demand responses to market order sell shocks are zero, the coefficients of the demand and supply functions are constant through time and the adjustment parameters are zero

\[ \gamma_s = 0, \lambda_s = 0, \delta_s = \delta, a_s = a, \eta_s = \eta, \text{ and } b_s = b. \]

Hence, the price process becomes

\[
\ln p(t) = \frac{1}{\delta + a} U(t) - \frac{1}{\eta + b} V(t) + \ln p(0), \text{ while}
\]

\[
Q(t) = \frac{a}{\delta + a} U(t) + \frac{b}{\eta + b} V(t).
\]

In general, the price process is the difference of two increasing pure jump processes and is of finite variation. By virtue of this characteristic of bounded variation and consistent with the absence of arbitrage opportunities, the price process must be a semimartingale. Geman et al., cite the result of Monroe (1978) who has shown that every semimartingale can be written as a Brownian motion \( W(t) \) evaluated at a random time change \( T(t) \), where \( T(t) \) is an increasing stochastic process:

[..m]ore formally, it is required that there exists a filtration with respect to which the process \( W(t) \) is adapted, and that \( T(t) \) is an increasing sequence of stopping times adapted to this filtration (Geman et al., 1998, fn. 9, p. 14),

such that
\[
\ln p(t) = \ln p(0) + W(T(t))
\]

Moreover, this process, in being the difference of two pure jump processes

\[
X_1(t) = \sum_{s \leq t} \Phi^\ast(\Delta U(s), s), \quad \text{and}
\]

\[
X_2(t) = \sum_{s \leq t} \Phi^\ast(\Delta V(s), s)
\]

is a pure jump process of finite variation

The authors also note, in consequence, that \( Q(t) \) is a pure jump process of finite variation and the volume transacted between two time points \( t_1 < t_2 \), \( V(t_1, t_2) \) is determined by

\[
V(t_1, t_2) = Q(t_2) - Q(t_1)
\]

This equilibrium approach has the virtue reducing the study of complex supply and demand interactions down to a question of how best to model the study of time changes for Brownian motion. For such a time change, the diffusion component must be zero, otherwise it will not be an increasing process, hence, the continuous time changes are locally deterministic. Ané and Geman (1997) have confirmed that when the highly non-normal returns for the FTSE100 index are measured per unit trade they become normally distributed.

Geman et al., consider a variety of candidate distributions for the log price relative

\[
X(t) = \ln p(t)/p(0)
\]

which are pure jump processes characterised by independent and identically distributed increments over non-overlapping intervals of regular length. They recognise the econometric evidence in favour of time inhomogeneity in both the statistical and risk neutral price process, but argue that time homogeneity is a reasonable starting point, and that inhomogeneity can be accommodated in a form suited to particular applications. They consider processes characterised by their Lévy densities \( k(x) \)—defining the arrival rates of price jumps of size \( x \)—which can be uniquely identified by the specific decomposition of their characteristic functions in accordance with the Lévy Khintchine theorem, which asserts that

\footnote{In Platen’s approach (1998), the state variables are governed by a Wiener process, while in Geman et al’s work, they are represented by a cumulative increasing random process. To test the overall model, Platen considers a parametric class of Gaussian mixture distributions comprehensively examined in Barndorff-Nielsen (1977, 1978)—the symmetric generalised hyperbolic class—which contains the Student \( t \) distribution as a special case. Geman et al., also consider the normal inverse Gaussian distribution as a possible model for the stock price process. They contend that this process,

“…may also be represented as a time-changed Brownian motion, where the time-change \( T(t) \) is the first passage time of another independent Brownian motion with drift to the level \( t \)…An interesting question from the perspective of this paper is whether such a process may be represented as the difference of two increasing processes that constitute the price responses to demand and supply shocks, respectively. However, this is not possible as the normal inverse Gaussian is one of infinite variation.” (Geman et al., p. 44)

Nevertheless, they suggest that it may be approximated by such a difference process “…by ignoring jumps of absolute size below \( \varepsilon \) and then letting \( \varepsilon \) tend to zero.”}
which for a Brownian motion with drift
\[ B(t) = \theta t + \sigma W(t) \]
evaluated at an independent random time \( T(t) \), with characteristic function
given by the Levy measure \( \tilde{k}(x) \), such that
\[ E[\exp(iuX(t))] = \exp \left( \int_{-\infty}^{\infty} (e^{iu} - 1)\tilde{k}(x)dx \right) \]

enables the time-changed Brownian motion \( X(t) = B(T(t)) \) to be
determined as follows
\[ E[\exp(iuB(T(t)))] = E[\exp(iu\theta T(t) + \sigma W(T(t)))] \]
\[ = E[\exp((iu\theta - \sigma^2 u^2/2)T(t))] \]
\[ = \exp \left( \int_{0}^{\infty} (e^{iu\theta - \sigma^2 u^2/2} - 1)\tilde{k}(x)dx \right) \]
\[ = \exp \left( \int_{-\infty}^{\infty} (e^{iu} - 1)\tilde{k}(x)dx \right) \]

In other words, the Lévy measure for the time change \( \tilde{k} \), can be calculated given the arrival rate Lévy measure \( k \), by solving for \( \tilde{k} \) in the last equality appearing above. When the time-change is no longer independent the calculation becomes more complicated.

Applying this theorem, Geman et al., solve for the characteristic function of the log price relative under the assumption that the density function of the prevailing sell and buy orders is given by the reflected normal, exponential, and gamma density functions. In the case of the reflected normal density function, they show that the time change “…just counts the number of all the demand and supply shocks, ignoring the magnitude of the shocks.” (p.21). However, the volatility is scaled to reflect price sensitivity. For exponential jump sizes, the time change reflects size-weighted cumulative order arrivals. Whereas for gamma density functions, the time change is shown to be related to the level of excess demand.

Lévy densities possess the property of monotonicity: jumps of larger sizes are postulated to have lower arrival rates than jumps of smaller sizes. This requires the derivative of differentiable density functions to be negative for positive jump sizes and positive for negative jump sizes. This property can be strengthened to complete monotonicity which requires that derivatives of the same order have the same sign. Geman et al., show that the time change for this class of completely monotone densities aggregates the counting of arrival orders with a mean arrival rate of \((1/a)\) and of size \((y)\) to arrive a clock-tick of duration \(a\exp(-a(a - 1)y))\). In this way they encompass the pre-existing literature relating price change to economic activity as measured by either the number or size of orders.

From the analysis of Boulding’s arguments about a liquidity preference theory of market prices, it can be seen that changes in the degree of uncertainty aversion will influence both the level and diversity of null prices, differentially across the liquidity spectrum of financial assets. When translated back into time-changed Brownian motion representations of financial market equilibrium, it must then be acknowledged that business activity will be influenced differentially for different classes of asset in a manner that may be impossible to capture within a model in which the state variables governing the frequency and average
value of order arrivals are assumed to follow a stable cumulative Poisson (as in Geman et al., 1998) or standard diffusion model (as in Platen, 1998). This position is argued by Juniper (2002).

**Risk-Sensitive and Robust Control Theory**

Elliott et al., (1995) introduce methods that have been developed and adopted by engineering control theorists and signal processing theorists for both the estimation and control of Hidden Markov models. These techniques have a two-fold advantage. First, a familiarity with these techniques afford access to an extensive body of literature spanning the fields of mathematical statistics, electrical engineering and signal processing broader than can be found within the standard econometrics literature. Second, the techniques introduced are similar to, though more basic than, those adopted for the robust and risk-sensitive control and filtering of more complex linear and non-linear diffusion processes (Elliott et al, 1995; James and Baras, 1995). For instance, techniques of dynamic programming, which employ operators derived from applications of Girsanov’s theorem, underpin the estimation and control of non-Gaussian stochastic processes, including Lévy processes and those characterized by multifractional Brownian motion (Helge et al., 1996).

Both Elliott et al., (1995) and Boel, James and Petersen (1997) draw on Dupuis and Ellis’s (1997) characterization of the duality between free energy and relative entropy to construct error bounds for risk-sensitive filters (also see Andersen, Hansen and Sargent, 1999). They assume that the true probability model is fixed but unknown, and that the estimation procedure makes use of a fixed nominal model. They show that the resulting error bound for the risk-sensitive filter is the sum of two terms, the first of which coincides with an upper bound on the error one would obtain if one knew exactly the underlying probability model, while the second term is a measure of the distance between the true and design probability models.

In the mathematical finance literature (e.g. Stutzer, 1995; Chan and van der Hoek, 2001) it is well known that the use of relative entropy (MinxEnt) to estimate risk-neutral probabilities, Gibbs state price probability densities, or equivalent martingale measures yields the same set of results as those obtained under the assumption that a representative investor determining the composition of their optimal portfolio has preferences associated with a constant absolute risk aversion utility function with a constant of proportionality equal to $-1/a$ (where $a$ is the coefficient of absolute risk aversion). It is hardly surprising; therefore, that the applications of the MinxEnt principle to problems of discrete choice yield results that can be related to underlying utility theory. Stutzer (1995, pp. 376-378) provides three additional interpretations of the state price density function based on: a quasi-maximum likelihood, a minimum information bound, and a Bayesian interpretation that is similar to that embodied in Laplace’s principle.

However, in financial applications of risk-sensitive control theory the object of study is uncertainty aversion rather than incompleteness in markets. The property of *bounded subadditivity*, which characterizes Cumulative Prospect Theory, one of the better known models of non-expected utility theory, has recently been extended by Tversky and Wakker (1995), to account for decision-making under uncertainty. The authors interpret this condition as implying that an event has a greater impact when it turns impossibility into possibility or possibility into certainty, than when it merely makes a possibility more likely (Tversky and Wakker, 1995, p. 1264).

Gilboa and Schmeidler (1989) have established the mathematical equivalence between two capacity-based representations of uncertainty aversion: the first of these entailing the use of *sub-additive probabilities*, and the second, involving max-min optimization within a *multiple-priors* setting. In a multiple-priors context, ambiguity aversion arises when agent’s relevant probabilistic beliefs are given by a set of probability measures rather than a singleton distribution. In characterizing the optimal rules in this context, researchers assume that economic agents adopt an intertemporal max-min expected utility approach: in a game –
theoretic context, nature is presumed to be malicious in maximizing a penalty function through the choice of a particular probability density from within the range of permissible distributions. The agent is then presumed to minimize the same penalty function through the choice of a (sub-optimal) control law and filter. These rules are designed to protect the agent against unfavourable probabilistic structures in the financial environment. In this control theoretic context, the duality between free energy and relative entropy applies to the stochastic uncertainty constraint, which accounts for (multiplicative) model uncertainty, observation error, and (typically non-Gaussian) perturbation.

Marinacci (1999) outlines a set of behavioural considerations that might motivate an approach to decision-making predicated on uncertainty aversion, while in Epstein and Schneider (2001), an axiomatic basis for uncertainty aversion has been constructed deploying a discrete-time, multiple-priors, recursive utility framework. A continuous-time variant is discussed in Chen and Epstein (2000). Also, see the debate between Epstein and Schneider (2001), and Hansen et al. (2001) over the precise nature of the relationship holding between risk-sensitive penalty functions and multiple-priors forms of generalized utility. Significantly, Grant and Quiggin have shown how Epstein and Zhang’s (2001) definition of ‘ambiguous events’ can be used to define ambiguity aversion over preference relations in “a solely preference-based and model-free manner” (Grant and Quiggin, 2002, p. 2).

Tsallis Entropy as an Approach to Anomalous Diffusion and NonExpected Utility Theory

Boltzmann-Shannon-Gibbs (BSG) entropy is defined by (Abe, 2000):

\[ S(p_1, p_2, \cdots, p_W) = -k \sum_{i=1}^{W} p_i \ln p_i. \]

Tsallis entropy is defined by:

\[ S_q(p_1, p_2, \cdots, p_W) = \frac{1}{1-q} \left[ \sum_{i=1}^{W} (p_i)^q - 1 \right] (q > 0), \]

In this generalization of BSG-entropy, the \( q \) parameter represents the degree of nonextensivity. Tsallis’s nonextensive measure provides a useful ansatz for the calculation of solutions to certain nonlinear partial differential equations (see the section on finance applications below). Under appropriate constraints the maximization of Tsallis entropy also yields exact time-dependent solutions for a family of non-linear Fokker-Planck equations representing anomalous diffusion and certain self-organizing phenomena. These equations are characterized by a diffusion term depending on the power of the probability density.

In the 20s and 30s, Lévy was concerned with the question of how to represent a situation of fractal scaling where the sum of identically random distributed variables has the same probability distribution as any one of the terms in the sum. The resulting distributions are now called Lévy’s stable laws (Shlesinger et al., 1987, p. 1100).

Drawing on initial work by Sainty (1992), Jumarie (2000, Chapters 6 and 7) sets out a mathematically simpler construction of complex-valued fractional Brownian motion (\( \mathcal{C}^\left((fBm)\right) \)), conceived as the limit of random walks in the complex plane.

In contrast to a conventional random walk, for which large step lengths are (exponentially) rare, a Lévy flight is a random walk whose step length occurs with a power law frequency (Gupta & Campanha, 2002, p. 531). Thus Lévy flights have infinite variance. In real systems the variance of a stationary process is finite. Therefore, to describe such systems using Lévy flight processes, some kind of arbitrary cut-off must be imposed. Early research in this vein by Mantegna and Stanley (1995) deployed a truncated Lévy flight process. More recent developments (Gupta and Campanha, 2000, 2002) allow for the gradual elimination of large step sizes by using an exponential, capacity-related, cut-off term. The resulting gradually truncated
Lévy distributions (GTLDs) approach the Gaussian distribution at relatively low-frequencies, but at high frequencies gives rise to a power-law distribution.

While GTLDs are based on positive feedback and physical limitations, Tsallis statistics are based on generalized thermodynamic considerations. Nevertheless, both statistics yield almost the same distribution. For this reason, Gupta and Campanha (2002, p. 385) speculate that the parameters of the GTLD are related to the \( q \) parameter because the limit that arises is due to similar thermodynamical or other natural requirements.

For statistically independent systems \( A \) and \( B \), under Tsallis entropy, it is well known that pseudoadditivity obtains, namely:

\[
S_q[A, B] = S_q[A] + S_q[B] + (1-q)S_q[A]S_q[B] \quad \text{(see Di Sisto et al, 1999)}.
\]

Another characterization of pseudoadditivity for systems \( A \) and \( B \) is:

\[
S_q(p_1, p_2, \ldots, p_W) = \left( p_{L,M} \right) + \left( p_{L} \right)^q \sum_{i=1}^W p_i \times \left( \frac{p_L}{p_L}, p_M, p_{2L}, \ldots, p_W \right) \left( p_L \right) + \left( p_{M} \right)^q \sum_{i=1}^W p_i \times \left( \frac{p_M}{p_M}, p_L, p_{2M}, \ldots, p_W \right) \left( p_M \right)
\]

where, \( p_L = \sum_{i=W_L+1}^W p_i \), \( p_M = \sum_{i=W_M+1}^W p_i \), with \( W_L + W_M = W \), and the respective sets \( \{p_L/p_L\} \) and \( \{p_M/p_M\} \) are conditional probabilities. Because \( p_i^q > p_i \leftrightarrow q < 1 \), \( p_i^q < p_i \leftrightarrow q > 1 \), superadditivity can be seen to privilege rarely events while subadditivity privileges frequent events (Tsallis et al, 1998, p.535). While subadditivity is familiar to theoretical physicists important applications have also occurred in quantitative finance and the theory of decision-making under uncertainty (see Schmeidler, 1989). Tsallis et al (2003) comment on the relationship between this property of pseudoadditivity and Cumulative Prospects Theory (CPT)—Kahneman and Tversky’s (1979) model of non-expected utility.

Abe (2000) shows how the Shannon-Khinchin axioms for Boltzmann-Shannon entropy can be modified to accommodate Tsallis entropy. His paper establishes that a quantity satisfying the transformed axioms is uniquely equal to Tsallis entropy. Thus, his uniqueness result represents a natural generalization of the Shannon-Khinchin result for Boltzmann-Shannon entropy by establishing a parallelism with the original axioms. Tsallis et al (1995) further argue that the ubiquity and robustness of the Levy distribution follow naturally from the generalized central-limit theorem, which applies to convolutions of distributions. Significantly, they further demonstrate that that Tsallis entropy generalizes the traditional inverse relationship known to hold between Boltzmann-Shannon entropy and the exponential function.

Abe (1997) demonstrates that Tsallis entropy can be interpreted using Jackson’s generalized differential operator. While Jackson’s operator “tests” the function \( f(x) \) under dilation, the usual derivative tests it under translation. This feature explains the usefulness of Tsallis entropy for describing chaotic systems with multifractal characteristics. Abe and Turner (2005) show how the assumptions made by Einstein in his classic derivation of Brownian motion can be relaxed (specifically the assumption relating to the existence of the second moment of the distribution is replaced by one assuming that the distribution has a divergent second moment whose characteristic function is given by a ‘stretched exponential form’), thus permitting the resulting diffusion equation to be solved (using the techniques of fractional calculus) to yield the Levy distribution. Di Sisto et al, (p. 597) demonstrate the specific relationship between Tsallis entropy is related to Renyi entropy. Under appropriate moment constraints over the first and second moments of the distribution, Boltzmann-Shannon entropy can be used to derive the normal distribution. De Souza & Tsallis (1997) show that under slightly modified moments constraints (which take into account the divergence of the second moment, Tsallis entropy can be used to derive the Students-t distribution.
Applications of Tsallis Entropy to Finance Theory

For a Lévy process, the correlation function defined over the number of transactions occurring within a given time interval displays a power-law decay. Moreover, the variance of the individual price changes due to the transactions occurring within a specified interval of time is, in turn, characterized by a distribution possessing a power-law decay. However, the correlation function for the variance of the individual price changes exhibits a short-range exponential decay. The existence of these characteristics can be confirmed or disconfirmed using de-trended fluctuation analysis and a graphical analysis of the relevant empirical distributions (Plerou et al., 2000; Gopikrishnan et al., 2000; Gupta and Campanha, 2002).

Despite the name, the basic techniques for de-trended fluctuation analysis are fairly straightforward. First, the time-series is differenced around a trend rate of growth estimated over the whole horizon for which data is available. Second, the time-series is divided into a sequence of windows defined by sub-intervals of a given duration. A series of simple trend regressions are then performed within each window. The root-mean-squared-error of the deviations of the process around each trend regression are then calculated and summed up. The length of the sub-intervals is then changed and the root mean squared error is calculated again. The resulting root-mean-squared-errors are then graphed against each of the selected sub-interval lengths (say, 10 minutes, 60 minutes, 200 minutes, 1000 minutes).

Based on results set out in (Borland, 1998), Lisa Borland (2002) examines an option pricing model based on a non-linear Fokker-Planck diffusion equation of the kind proposed for those systems with correlated anomalous diffusion (i.e. Lévy superdiffusion or subdiffusion) as observed in plasma flow, porous media, surface growth, and NMR relaxometry of liquids in porous glasses). This model provides an alternative to those using a diffusion equation with fractional derivatives (i.e. Lévy diffusion). Tsallis (non-extensive) entropy, which is closely related to the scaling properties of multifractal attractors) has been used to provide a thermostatistical basis for Lévy-type anomalous diffusion. Where the standard Fokker-Planck and Ito-Langevin equations for Gaussian diffusion processes is given by (Borland, 1998, pp. 6634-6):

\[
\frac{df}{dt} = -\frac{d}{dx}[K(x,t)f] + \frac{1}{2} \frac{d^2}{dx^2} [g^2(x,t)f]
\]

and

\[
\frac{dx}{dt} = s(x,t) + g(x,t)h(t),
\]

their non-linear counterparts are,

\[
\frac{df^{\mu}}{dt} = -\frac{d}{dx}[Kf^{\mu}] + Q \frac{d^2}{dx^2} [f^{\nu}]
\]

and

\[
\frac{dx}{dt} = K(x) + \sqrt{Q} f(x,t)^{\nu/2} \eta(t)
\]

Borland (1998) focuses on the \( \mu = 1 \) case, after demonstrating that the results for the more general case could be mapped onto the former case by introducing a new variable \( \tilde{f} = f^{\mu} \), and using \( \tilde{\nu} = \nu/\mu \).

The form that satisfied the stationary nonlinear Fokker-Planck equation is,
\[ f_q(x) = \frac{1}{Z_q} [1 - \beta (1 - q) V(x)]^{1/(1-q)}, \]

\[ V(x) = \int K(x') dx', \]

\[ \beta = 2 \frac{Z_q^{q-1}}{Q (2 - q)}, \]

\[ \nu = 2 - q. \]

Furthermore, this distribution maximizes the generalized Tsallis entropy, which for the continuous-rather than discrete-range case, has the form,

\[ S_q = \frac{1 - \int dx [f(x)]^q}{q - 1}. \]

As with discrete-range distributions, this expression reduces to Boltzmann-Gibbs entropy \( S = -\int f \ln f dx \) in the limit \( q \to 1 \). And again, the degree of non-additivity is quantified by the Tsallis index \( q \). Because \( f \) is a positive quantity, we must impose \( f(x) = 0 \) if the term in the square brackets in the above expression for the solution becomes negative. This is known as the Tsallis cut-off and occurs when,

\[ \frac{1 - q}{2 - q} > \frac{Q}{2 Z_q^{q-1} V(x)}. \]

Within the range \( 1 \leq q \leq 5/3 \), positive tails and finite variances are found.

Borland shows that the process satisfies the scaling relationship,

\[ \langle x(bt)^2 \rangle = b^{2/(3-q)} \langle x(t)^2 \rangle, \text{ for } t \to \infty. \]

The same result obtains for the more general case where \( \mu \neq 1 \), except that now we have (p. 57),

\[ q = 2 - \frac{\nu}{\mu}. \]

The above model is characterized by the fact that the stochastic force term in the Fokker-Planck equation depends on the probability distribution for \( f \), which in turn gives rise to an \( f \)-dependent Ito-Langevin equation. Borland interprets the stochastic trajectories described by the latter as the motion of particle within a potential well defined by \( V \), which is knocked around by the stochastic force envisaged as a series of random kicks. In the standard Brownian motion case (\( \nu = 1 \)), the random kicks are uniform and do not depend on where the particle happens to be. In the \( \nu \neq 1 \) cases, the size of the kick will either be larger or smaller in highly frequented regions of the potential well. This biases the ergodic behavior of the system, and could transform the phase space into one governed by multifractal attractors.

In their closely related analysis Plerou et al., interpret the fluctuations in the volume of transactions as a reflection of ‘avalanches’ where trades beget new trades (as in the work of Lux and Marchesi, 1999; and Bouchard and Cont, 1998), whereas they relate the fluctuations in price volatility to (i) the level of liquidity in the market; (ii) the risk-aversion of participants, and (iii) uncertainty about the fundamental value of the asset.
Borland compares the process defined by this nonlinear Fokker-Planck equation with fractional Brownian motion, where the stochastic path is described by (p. 6641),

\[ x(t) = \Gamma \int_0^t (t - \tau)^{H-3/2} d\tau , \]

and the process scales as,

\[ \langle x(bt)^2 \rangle = b^{2H} \langle x(t)^2 \rangle . \]

Here, \( H \) is the Hurst parameter defined by,

\[ \frac{R}{S} = \left( \frac{\tau}{2} \right)^H , \]

with \( R = \max_{1 \leq i \leq T} X(t, \tau) - \min_{1 \leq i \leq T} X(t, \tau) , \)

and \( X(t, \tau) = \sum_{i=1}^{\tau} [\zeta(t) - \langle \zeta \rangle \tau] \)

with \( \langle \zeta \rangle \tau = \frac{1}{\tau} \sum_{t=1}^{\tau} \zeta(t) , \)

and, \( S = \left( \frac{1}{\tau} \sum_{t=1}^{\tau} [\zeta(t) - \langle \zeta \rangle \tau]^2 \right)^{1/2} , \)

The process generated by the Ito-Langenvin equation scales in time as a fractional Brownian motion process where the relationship between the Tsallis and Hurst parameters is given by,

\[ qH = 3/2 . \]

Given the valid range for the Hurst parameter, this expression only makes sense for \(-\infty < q < 2\). If a process diffuses with scale defined by \( \langle x(t)^2 \rangle \propto t^\gamma \) with \( 0 < \gamma < 2 \) it may either conform a fractional Brownian motion process or one characterized by anomalous diffusion. However, for a rage of feasible \( q \) parameter values ranging from 1 to 4, Borland (1998, p. 6641) establishes that the Hurst exponent has the value 0.5, reflecting the fact that the process has no memory. Borland (1998, p. 6636) also considers coloured noise as given by a stochastic volatility process. Significantly, she demonstrates that the nonlinear terms in the diffusion term of the Fokker-Planck equation cannot be generated by coloured noise.

Borland (2002) examines the log-price relative of 10 high-volume NASDAQ stocks, showing that Tsallis distribution provides a good fit to the data. She then assumes that log returns follow a simplified version of the nonlinear Fokker-Planck equation (setting \( K = 0 \)), obtaining a generalized Wiener distribution that reverts to the Gaussian case when \( q \to 1 \). Under the efficient markets assumption, the discounted log return process will be a Martingale. Using Girsanov’s theorem, Borland derives the requisite Radon-Nikodym derivative and the appropriate Equivalent Martingale Measure, enabling her to derive the formula for the price of a European option defined over the resulting stochastic process. Significantly, she establishes that the implied volatility for \( q = 1.5 \), matches that observed in the market for British Pound futures and S&P500 futures.


